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BRAUER GROUPS ARE NOT CHARACTERIZED BY ULM INVARIANTS

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ABSTRACT

Two important invariants of a field F are its Brauer group $B(F)$ and its character group $X(F)$. If F is countable, these are countable abelian torsion groups, and so are determined by their Ulm invariants. We show here that Ulm's invariants do not determine Brauer groups or character groups of uncountable fields. An essential tool, which is entirely group theoretic in nature, is a fact about ultraproducts of torsion groups.

1. Introduction

Fein and Schacher studied Ulm invariants of the Brauer group and character group of a field in [1]–[4] in an effort to characterize these groups. This research led to the following theorem (p. 532 of [4]):

THEOREM A. *If E is a field which is finitely generated over a global field, E_n the pure function field over E in n variables, and $B(E_n)$ the Brauer group of E_n, then $B(E_n) \cong B(E_m)$ provided n, $m \geqq 1$.*

The proof of Theorem A proceeds from the observation that the fields in question are countable, and so their Brauer groups and character groups are countable as well, and thus are characterized by their Ulm invariants. This leaves open the question of whether Brauer groups and character groups of uncountable fields are necessarily characterized by their Ulm invariants. We show in this paper that they are not.

For the most part we will keep the notation and terminology of [1]–[4]; we outline below some of the definitions and notions which arise.

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Let G be an abelian torsion group; we will always consider G as an additive group. Our standard reference for abelian group theory will be [7]; all results we will need on infinite abelian groups are proved there. If p is a prime, we set G_p = the p-primary subgroup of G. Then one has the direct sum decomposition (Theorem 1 of [7]):

$$G \cong \bigoplus_p G_p$$

where the direct sum runs over all primes p. To characterize G, it is enough to characterize the primary components G_p. Theorem 3 of [7] gives

$$G_p \cong D_p \oplus R_p$$

where D_p is the maximal divisible subgroup of G_p, and R_p is a "reduced" p-group, i.e. R_p contains no divisible subgroups, and is unique up to isomorphism. D_p as a divisible p-group is a direct sum of copies of the group $\mathbf{Z}(p^\infty)$, and so is characterized by its rank — the number of copies of $\mathbf{Z}(p^\infty)$. We write $r_p(G)$ for this divisible rank.

For invariants of R_p, we set $P = \{x \in R_p \mid px = 0\}$. For any ordinal λ, we define inductively:

$$R_p(0) = R_p, \quad R_p(\lambda+1) = pR_p(\lambda), \quad \text{and} \quad R_p(\lambda) = \bigcap_{\beta<\lambda} R_p(\beta)$$
$$\text{if } \lambda \text{ is a limit ordinal.}$$

The smallest λ with $R_p(\lambda) = 0$ is called the Ulm length of R_p, and written $l_p(G)$. If $P_\lambda = P \cap R_p(\lambda)$, then the λ-th Ulm invariant of G at p is

$$U_p(\lambda, G) = [P_\lambda / P_{\lambda+1} \colon \mathbf{Z}/p\mathbf{Z}]$$
$$= \text{the dimension of } P_\lambda/P_{\lambda+1} \text{ over the field of } p \text{ elements.}$$

Ulm's theorem ([7], Theorem 14]) says that the invariants $U_p(\lambda, G)$ are a complete set of invariants for R_p when R_p is countable. In any case, these dimensions are invariants of G.

By the Ulm invariants for G we will mean the entire set of invariants $r_p(G)$ and $U_p(\lambda, G)$, $0 \leqq \lambda < l_p(G)$, for all primes p.

In Section 3 we construct two fields K and L for which the Brauer groups $B(K)$ and $B(L)$ have identical Ulm invariants but are not isomorphic. In Section 4 we prove similar results for character groups. All of these constructions depend on a (essentially known) result about ultraproducts of p-groups; for completeness we prove this result in Section 2.

In all that follows, we will let ω denote the first infinite ordinal and also the corresponding cardinal, and $\mathscr{C}$ the cardinal 2^{ω}. If g is an element of an abelian group G we denote by $\langle g\rangle$ the cyclic group generated by g. $T(G)$ will denote the torsion subgroup of any abelian group G. Our symbol for the empty set will be $\varnothing$.

2. Ultraproducts of p-groups

The following notation will be in force through this section:
$I=\{1,2,3,\ldots,\ldots\}$ the set of positive integers,
p a fixed prime,
$\mathscr{F}$ a non-principal ultrafilter on I,
G_i for each $i\in I$, a reduced abelian p-group,
$G=\Pi_{\mathscr{F}}G_i$ the ultraproduct of the G_i,
$T=T(G)$ the torsion subgroup of G.

THEOREM 1. (1) *$T=D\oplus T_0$ where D is the maximal divisible subgroup of T, and T_0 is a reduced p-group of Ulm length $\leqq\omega$.*

(2) *If all the G_i are isomorphic to an unbounded p-group H, then T_0 is not a direct sum of cyclic groups.*

PROOF. A quick proof of this would go as follows: G is ω_1-equationally compact and therefore algebraically compact. The structure theory for algebraically compact groups then gives the desired result on T and T_0. For details on this approach see Fuchs [6], Chapter 7 (see also Eklof [0]). We give here a proof which is self-contained modulo Kaplansky [7].

(1) Since T and T_0 are clearly p-groups, it will suffice to show that for each $g\in T$, either g lies in a divisible subgroup of T or g has finite p-height $h(g)$ (see [7], Sections 9, 10).

Choose a representative tuple (g_i) for g. For each integer $n\geqq 0$, let $I_n=\{i\mid h(g_i)=n\}$. If $I_n\in\mathscr{F}$ for some n, then clearly $h(g)=n$. Thus we may assume all $I_n\notin\mathscr{F}$. The union of the I_n's may not exhaust I, so we choose a partition $I=J_0\cup J_1\cup\cdots\cup J_n\cdots$ where:

(i) $I_n\subset J_n$ for each n,

(ii) $J_n\notin\mathscr{F}$ for each n,

(iii) $J_n\cap J_m=\varnothing$ if $n\neq m$.

Thus, for each $i\in I$, if $i\in J_{n(i)}$, we have $h(g_i)\geqq n(i)$. For such i we choose elements $h_i^{(1)},h_i^{(2)},\ldots,h_i^{n(i)}$ with $ph_i^{(1)}=g_i$, $ph_i^{(2)}=h_i^{(1)},\ldots$ etc. Now define elements $h^{(1)},h^{(2)},\ldots,h^{(k)},\ldots$ of T as follows: for each k, $h^{(k)}$ is represented by the tuple $(h_i^{(k)})$. (Since $\{i\in I\mid h(g_i)<k\}\notin\mathscr{F}$, we may let the $h_i^{(k)}$ be arbitrary if

$k > n(i)$.) Then the elements $g, h^{(1)}, h^{(2)}, \ldots$ generate a copy of $\mathbf{Z}(p^\infty)$ in T, since $p(h_i^{(k+1)}) = h_i^{(k)}$ for those i with $h(g_i) \geqq k+1$, and this set is in $\mathcal{F}$, giving $ph^{(k+1)} = h^{(k)}$ and $ph^{(1)} = g$.

The proof of (2) in Theorem 1 will require several reductions. For each i, let B_i be a basic subgroup of G_i, so:

(a) G_i/B_i is divisible,

(b) B_i is a direct sum of cyclic groups,

(c) B_i is pure in G_i.

The B_i exist by [7, Lemma 21]. (It should be noted that Lemma 21 of [7] is stated under the hypothesis that G_i has no elements of infinite height, but the proof does not require this. For an alternate proof see Fuchs [6], Theorem 32.3.) Set $B = T(\Pi_{\mathcal{F}} B_i)$ considered as a subgroup of T.

Claim 1. B is a pure subgroup of T, and $B + D = T$.

Claim 2. If $g_i \in G_i$, we can write $g_i = b_i + x_i$ where $b_i \in B_i$, $o(b_i) \leqq o(g_i)$, $o(x_i) \leqq o(g_i)$, and $h(x_i) \geqq i$.

We prove Claim 2 first: say $g \in G_i$ has order p^n, and write $\bar{g}$ for its image (mod B_i). Then $o(\bar{g}) = p^m$ for $m \leqq n$, and by (a) we have $\bar{g} = p^i\bar{h}$ where $\bar{h} \in G_i/B_i$ has order p^{m+i}. Since $B_i \subset G_i$ is pure, there is an element $y \in G_i$ with $\bar{y} = \bar{h}$ and $o(y) = o(\bar{h}) = p^{m+i}$ by [7, Lemma 1]. Set $x_i = p^i y$ and $b_i = g - x_i$, so $b_i \in B_i$ as $\bar{b}_i = \bar{g} - p^i\bar{h} = 0$. Now $h(x_i) \geqq i$ since $x_i = p^i y$, and $o(x_i) = p^m \Rightarrow o(b_i) \leqq \max(o(g), o(x_i)) = o(g)$.

We can now prove Claim 1: suppose $g \in T$. It is clear that B is pure in T since each B_i is pure in G_i. Let (g_i) be a representative tuple for g, and $g_i = b_i + x_i$ the decomposition of Claim 2. The g_i have bounded order (on a set in $\mathcal{F}$) since $g \in T$, and so b_i and x_i do also. Then $g = b + x$ where $b = (b_i)$, $x = (x_i)$, and x and b are both in T since their components have bounded order on a set in $\mathcal{F}$. Clearly $b \in B$, and (1) of Theorem 1 shows $x \in D$. Thus $B + D = T$.

Now consider the projection of B on the direct summand T_0 of T. This projection is surjective by Claim 1 and has kernel $B \cap D$. The elements of $B \cap D$ have infinite height in T, and so infinite height in B since $B \subset T$ is pure. Then these elements lie in a divisible subgroup of B by (1) applied to B, i.e. $B \cap D$ is divisible. It follows that $B = (B \cap D) \oplus T_0$, with $B \cap D$ the maximal divisible subgroup of B. Hence, without loss of generality, we may assume $B_i = G_i$, i.e. each G_i is a direct sum of cyclic groups.

We have not yet used the assumption that all G_i are isomorphic. We note, however, that some such hypothesis is crucial: if the supports of the Ulm invariants of the G_i are pairwise disjoint, then T_0 will be trivial, i.e. $T = D$ will be divisible. Say now all $G_i \cong H$, where H is an unbounded direct sum of cyclic p-groups.

It will be sufficient to consider the case when all Ulm invariants of H are 0 or 1. This follows since any H not of bounded order contains a direct summand H' of this sort. Taking G'_i to be the corresponding subgroup of G_i, then $T(\Pi_{\mathcal{F}} G'_i)$ will be a direct summand of T, and its reduced group T'_0 will be a subgroup of T_0. Since subgroups of direct sums of cyclic groups are direct sums of cyclic groups ([7, Theorem 3]), it will suffice for our purpose to show T'_0 is not a direct sum of cyclic groups.

The argument when all Ulm invariants are 1 is typical, so we consider this case. We have $G_i = \Sigma_{j=1}^{\infty} G_{ij}$ where $G_{ij} = \langle g_{ij} \rangle$ and g_{ij} has order p^j. To show that T_0 is not a direct sum of cyclic groups, it will suffice to find a subgroup of T_0 which is not a direct sum of cyclic groups. The "standard" example of a p-group with no elements of infinite height which is not a direct sum of cyclic groups is the torsion completion of our H, i.e. $U = T(\Pi_{j=1}^{\infty}\langle h_j \rangle)$ where h_j has order p^j and we have taken the complete direct product. We will construct an embedding θ of U into T_0.

Let $h \in U$ with $h = (l_j h_j)$, so there is a bound on the orders of the $l_j h_j$, $l_j \in \mathbf{Z}$. We define $\theta(h)$ to be the image of the representative tuple (g_i) where:

$$\begin{aligned} g_1 &= l_1 g_{11}, \\ g_2 &= l_1 g_{21} + l_2 g_{22}, \\ &\vdots \\ g_i &= l_1 g_{i1} + l_2 g_{i2} + \cdots + l_i g_{ii}, \\ &\vdots \end{aligned}$$

The g_i have bounded order since $o(h_j) = o(g_{ij}) = p^j$, and so $(g_i) \in T$. It is easy to see that θ induces a homomorphism from U into T. Furthermore, θ is 1-1 since

$$\theta(h) = 0 \Rightarrow g_i = l_1 g_{i1} + \cdots + l_i g_{ii} = 0$$

for infinitely many i

$$\Rightarrow l_1 g_{i1} = l_2 g_{i2} = \cdots = l_i g_{ii} = 0 \Rightarrow l_1 h_1 = l_2 h_2 = \cdots = l_i h_i = 0 \Rightarrow h = 0.$$

It is straightforward to check that the image of θ is pure in T. Any element in $\theta(U) \cap D$ has infinite height in T, and so infinite height in $\theta(U)$. But U has no elements of infinite height, so $\theta(U)$ has no elements of infinite height. Therefore, $\theta(U) \cap D = 0$, so $\theta(U)$ is contained in an isomorphic copy of T_0, as required. This completes the proof of Theorem 1.

Two remarks are in order. We will need to know something about the Ulm invariants of T_0 in (2) of Theorem 1 when all $G_i \cong H$ and H is countable. In that case, we have:

(2.1) The n-th Ulm invariant of T_0 will coincide with that of H when $U_p(n, H)$ is finite and will otherwise be equal to $\mathscr{C} = 2^{\omega}$.

(2.2) Theorem 1 remains valid if all $G_i = K \oplus H$ where K is a divisible p-group and H an unbounded reduced p-group. It was easier to formulate the proof without worrying about K.

3. Applications to Brauer groups

Let K be a field of characteristic 0. For any integer $n \geqq 1$ we write $\varepsilon(n)$ for a primitive n-th root of unity over K. We denote, as in [1]–[4], $X(K)$ for the character group of K and $B(K)$ for the Brauer group of K. We are interested in how these groups are evaluated when K is an ultraproduct. For the general set-up, we have:

$I = \{1, 2, 3, \ldots\}$,

$\mathscr{F}$ a non-principal ultrafilter on I,

K_i, for each $i \in I$, a field of characteristic 0,

p a prime,

$K = \Pi_{\mathscr{F}} K_i$.

K is then a field of characteristic 0. Suppose $L \supset K$ is a field extension of dimension n. Then $L = \Pi_{\mathscr{F}} L_i$ where $L_i \supset K_i$ is an n-dimensional field extension of K_i for a set of i in $\mathscr{F}$. Say now $M \supset L$ is a cyclic extension of L of dimension p^a. Then $M = \Pi_{\mathscr{F}} M_i$ where $M_i \supset L_i$ is cyclic of dimension p^a for a set of i in $\mathscr{F}$. Conversely, the L_i and M_i determine L and M. Using the correspondence between cyclic extensions and characters of [1]–[4], we get:

$$X\Big(\prod_{\mathscr{F}} L_i\Big)_p \cong T\Big(\prod_{\mathscr{F}} X(L_i)_p\Big). \tag{3.1}$$

The situation for the Brauer group is more complicated. In general, $B(L)_p$ is not equal to $T(\Pi_{\mathscr{F}} B(L_i)_p)$. If, for instance, L_i supports a central division ring D_i of exponent p but index p^i, then $D = \Pi_{\mathscr{F}} D_i$ will be a division ring which is not finite-dimensional over L, and so does not represent a class in $B(L)_p$. However, the expected correspondence makes sense if the L_i have the additional property on central division algebras that exponent and index are equal; this applies if, for instance, the K_i are number fields. It holds also if the K_i are algebraic over the rational field $\mathbf{Q}$. To see this, let V be a field algebraic over $\mathbf{Q}$ and D a finite-dimensional central division algebra over V. Then D is of form $D = D_0 \otimes_{V_0} V$ where V_0 is a number field and D_0 is central over V_0. The equality of exponent and index for D follows from the equality of these invariants for D_0. We have:

(3.2) If exponent and index are equal for central division algebras over L_i, then $B(L)_p \cong T(\Pi_{\mathcal{F}} B(L_i)_p)$. This holds in particular if all L_i are algebraic over $\mathbf{Q}$.

Let $F = \mathbf{Q}_3(\varepsilon(8))$ where $\mathbf{Q}_3$ is the field of 3-adic numbers. We use F to define a set of integer invariants with which we will construct an algebraic extension k of $\mathbf{Q}$. Let f be the function from primes to positive integers given by: $f(2) = 3$, and for odd p, $f(p) =$ maximum m so that $F(\varepsilon(p)) = F(\varepsilon(p^m))$. Let $k_0 = \mathbf{Q}(\varepsilon(8))$. Note that 2^n-th roots of unity over any field containing k_0 become cyclic, and k_0 contains no other roots of unity than the elements of $\langle\varepsilon(8)\rangle$. If p is an odd prime and $m = f(p)$ as above, then $k_0(\varepsilon(p^m))$ decomposes as a product of fields:

$$k_0(\varepsilon(p^m)) = A(p) \otimes_{k_0} B(\rho) \quad \text{where } [A(p): k_0] = p - 1 \text{ and } [B(p): k_0] = p^{m-1}.$$

We define k to be the field obtained from k_0 by adjoining to k_0 all of the fields $B(p)$. The crucial property of k is:

$$(3.3) \qquad k(\varepsilon(p)) = k(\varepsilon(p^m)), \quad m = f(p), \quad \text{and} \quad k(\varepsilon(p) \neq k(\varepsilon(p^{m+1})).$$

We remark that we de not know whether k is in fact a number field. The point is those p with $f(p) > 1$ satisfy $3^{p-1} \equiv 1 \pmod{p^2}$, and only for these p is $B(p)$ non-trivial. It has been conjectured that this congruence has solutions for only finitely many p; the truth of this conjecture would imply only finitely many non-trivial $B(p)$, so k would be finite-dimensional over $\mathbf{Q}$. However, to our knowledge this conjecture remains a famous unsolved problem.

We are now ready to construct our example. For each $i \in I$, let $K_i = k$, where k is the algebraic extension of $\mathbf{Q}$ constructed above. Set $K = \Pi_{\mathcal{F}} K_i$. Then K is a field for which (3.1) and (3.2) apply. Set $L_1 = K(t)$ for t an indeterminate over K. Let $L_2 = F(t)$ where t is an indeterminate over $F = \mathbf{Q}_3(\sqrt{-1}) = \mathbf{Q}_3(\varepsilon(8))$.

THEOREM 2. *$B(L_1)$ and $B(L_2)$ have identical Ulm invariants at all primes p, but are not isomorphic. In fact, for any p, $B(L_2)_p$ is a direct sum of a divisible group and cyclic groups, whereas $B(L_1)_p$ has no such presentation.*

PROOF. If E is any field of characteristic 0, then by the Auslander–Brumer theorem (see e.g. [3, 2.1]):

$$(3.4) \qquad B(E(t))_p \cong B(E)_p \oplus \bigoplus_{|E|} (X(M)_p)$$

where M runs over all finite extensions of E, and each summand $X(M)$ occurs $|E|$ times, $|E|$ = cardinality of E.

It follows that $r_p(B(L_i)) = \mathscr{C}$ for $i = 1$ or 2. We fix a prime p, and must show that the reduced parts of these groups have matching Ulm invariants at p. First let $E = K$. Then $B(E)_p = T(\Pi_{\mathscr{F}} B(k)_p)$. Each $B(k)_p$ is divisible since k_0 has no real embeddings (see [5, Theorem 2]), and so $B(E)_p$ is divisible.

Each factor $X(M)_p$ on the right side of (3.4) is a divisible group plus a reduced group of Ulm length $\leqq \omega$ by (3.1) and Theorem 1. Moreover, every such Ulm invariant in $B(E(t))$ is $\mathscr{C}$ if it is non-0 by (3.4) since the cardinality of E is $\mathscr{C}$. Since $M(\varepsilon(p)) = M(\varepsilon(p^m))$, $m = f(p)$, we have $U_p(\lambda, X(M)) = 0$ for $0 \leqq \lambda \leqq f(p) - 2$ (if $f(p) \geqq 2$) by [4, Lemmas 6 and 7].

To find the invariants of $X(K)_p$ at finite ordinals, we must first find the invariants of $X(k)_p$. We have $U_p(\lambda, X(k)) = 0$ for any prime p if $0 \leqq \lambda \leqq f(p) - 2$ by (3.3) and [4, Lemmas 6, 7]. Let p be a prime, $m = f(p)$, and $n \geqq m - 1$ an integer. For $k_0 = \mathbf{Q}(\varepsilon(8))$ as before, we set $S = k_0$ if $p = 2$ or S = the unique subfield of k of dimension p^{m-1} over k_0 if p is odd. Then $S(\varepsilon(p)) = S(\varepsilon(p^m))$, and S is a number field. By [4, Lemma 8], $U_p(n, X(S)) = \omega$. We obtain k from S by a sequence of field extensions all of dimensions prime to p; by the restriction–corestriction argument of [1, Theorem 3], $U_p(n, X(\bar{S})) = \omega$ for any subfield $\bar{S}$, $S \subset \bar{S} \subset k$, $\bar{S}$ finite-dimensional over S. Passing to the limit, $U_p(n, X(k)) = \omega$. Then from (2.1) and (3.1), $U_p(n, X(K)) = \mathscr{C}$. For any finite extension M of K, $U_p(\lambda, X(M)) = 0$ for $\lambda \geqq \omega$ by (3.1) and Theorem 1. Putting these facts together, the right side of (3.4) gives these invariants for any prime p:

(A) $U_p(\lambda, B(L_1) = \mathscr{C}$ for $f(p) - 1 \leqq \lambda < \omega$, and
$U_p(\lambda, B(L_1)) = 0$ for $0 \leqq \lambda < f(p) - 1$.

(B) $U_p(\lambda, B(L_1)) = 0$ for $\lambda \geqq \omega$.

Now let $E = F(t)$, $F = \mathbf{Q}_3(\sqrt{-1}) = \mathbf{Q}_3(\varepsilon(8))$. If M is any finite extension of F, $X(M)$ is a direct sum of a divisible group and a finite group. Since $M(\varepsilon(p)) = M(\varepsilon(p^m))$, $m = f(p)$, we have exactly as above: $U_p(\lambda, X(M)) = 0$ for $0 \leqq \lambda \leqq f(p) - 2$. Also, $B(F) \cong \mathbf{Q}/\mathbf{Z}$ is divisible.

Now suppose p is fixed and $n \geqq f(p) - 1$. Suppose first $p \neq 3$. Let M be the field obtained by adjoining a p^{n+1}-th root of unity to F. Then, in the language of [4, Section 3], $\phi(M, p) = n + 1$. Suppose π is a prime element of M, and S_1 the cyclic extension of degree p^{n+1} over M obtained by adjoining a p^{n+1}-th root of π.

Let $S \subset S_1$ be the unique subfield of dimension p over M. Then the arguments of [4, Section 3] show that S/M corresponds to an element of order p in $X(M)$ which has height equal to n (since S/M is tamely ramified for $p \neq 3$). It follows that $U_p(\lambda, X(M)) \neq 0$ when $\lambda = n$. Thus, for $p \neq 3$, the right hand side of (3.4) has exactly the Ulm invariants prescribed by (A) and (B) above.

Now let $p = 3$. Note that $f(3) = 1$. Let M be the field obtained by adjoining a 3^t-th root of unity to F. Then by [4, p. 526] (see also H. Koch, *Galois Theorie der p-Erweiterungen*, Springer-Verlag, New York, 1970) $X(M)_3$ has a direct summand which is a cyclic group of order p^t. Thus $X(M)_3$ has an element of height $t-1$, and so $U_3(t-1, X(M)) \neq 0$. We conclude $U_3(\lambda, B(L_2)) = \mathscr{C}$ for $0 \leqq \lambda < \omega$, and so (A) and (B) hold for $p = 3$ also.

We have demonstrated that the $B(L_i)$ have identical Ulm invariants at all primes. We must show they are not isomorphic. Consider first L_2. Since every term $X(M)$ on the right side of (3.4) is a direct sum of divisible groups and cyclic groups in this case, and $B(E) \cong \mathbf{Q}/\mathbf{Z}$, we conclude $B(L_2)$ is a direct sum of divisible and cyclic groups. For L_1: Applying (2) of Theorem 1, (3.1), and [7, Theorem 3], we conclude $B(L_1)$ has no presentation as a direct sum of divisible and cyclic groups. Thus the $B(L_i)$ have distinct p-primary components for all primes p.

4. Applications to character groups

We need one further result about character groups of function fields. Suppose F is a field of characteristic 0 containing only finitely many p-power roots of unity, and t is an indeterminate over F. We assume F contains 4-th roots of unity if $p = 2$. Since we could not find any reference to Theorem 3 in the literature, we include a proof here.

THEOREM 3. *Under the hypotheses above,* $X(F(t))_p \cong X(F)_p \oplus A$, *where* A *is a direct sum of copies of* $\mathbf{Z}(p^\infty)$ *and cyclic* p*-groups.*

PROOF. For any $n \geqq 1$ let $X(n)$ denote the elements of exponent p^n in $X(F(t))_p$, and $Y(n)$ the elements of exponent p^n in $X(F)_p$. Then $X(F(t))_p$ (respectively $X(F)_p$) is the direct limit of the sequence

$$X(1) \subset X(2) \subset X(3) \subset \cdots \qquad (\text{respectively } Y(1) \subset Y(2) \subset Y(3) \subset \cdots).$$

There is a natural injection from $Y(n)$ into $X(n)$ — we identify $Y(n)$ with its image under this injection.

Henceforth we use the notation and terminology of [8]. Let $q = p^n$ and let ρ

be a primitive q-th root of unity over F. Set $E = F(\rho)$ and $s = [E: F]$. We assume $s > 1$; this will hold for sufficiently large n. The hypotheses guarantee that the Galois group of E/F is cyclic of order s generated by some element σ. If $s = p^k s'$ where $p \nmid s'$, then there is a unique subfield E' of E with $[E': F] = p^k$. This E' corresponds to a subgroup $Y'(n)$ of $Y(n)$. We first show that $Y(n)/Y'(n)$ is a summand of $X(n)/Y'(n)$.

By [8, Lemma 2.2] we may assume $\sigma(\rho) = \rho^m$ where $(m^s - 1)/q$ is prime to p. Let $K = E(t)$, and extend σ to K by $\sigma(t) = t$. Define $\phi: K^* \to K^*$ by

$$\phi(x) = x^{m^{s-1}}(\sigma(x))^{m^{s-2}}(\sigma^2(x))^{m^{s-3}} \cdots (\sigma^{s-1}(x)).$$

Then by [8, Theorem 2.3]

$$X(n)/Y'(n) \cong \phi(K^*)/\phi(K^*) \cap (K^*)^q \cong \phi(K^*) \cdot (K^*)^q/(K^*)^q,$$

i.e. $X(n)/Y'(n)$ is isomorphic to the subgroup of $K^*/(K^*)^q$ generated by images of elements of $\phi(K^*)$. Similarly

$$Y(n)/Y'(n) \cong \phi(E^*)/\phi(E^*) \cap (E^*)^q \cong \phi(E^*)(E^*)^q/(E^*)^q.$$

Now let $\mathcal{M}$ be the collection of monic irreducible polynomials in $F[t]$. For each $f \in \mathcal{M}$, let $\mathcal{A}_f$ be the collection of monic irreducible factors of f in $E[t]$. Then unique factorization gives the following decomposition of K^*:

$$K^* = E^* \oplus \sum_{f \in \mathcal{M}} \Big(\sum_{g \in \mathcal{A}_f} \langle g \rangle \Big). \tag{4.1}$$

Since both ϕ and "q-th power" respect the outer decomposition in (4.1), we obtain a corresponding decomposition of $X(n)/Y'(n)$:

$$X(n)/Y'(n) = Y(n)/Y'(n) \oplus \sum_f X(n,f) \tag{4.2}$$

where the $X(n,f)$ will be described below.

We now pass to the direct limit, recalling that taking direct limits is an exact functor, to obtain the short exact sequence

$$0 \to \varinjlim Y'(n) \to X(F(t))_p \to \Big[X(F)_p \big/ \varinjlim Y'(n) \Big] \oplus \sum_f \varinjlim X(n,f) \to 0. \tag{4.3}$$

(Here we use the fact that the injection of $X(n)$ into $X(n+1)$ induces, in (4.2), the obvious injection on the first factor — and, on the second factor, the map induced by the "pth power" map $K^*/(K^*)^{p^n} \to K_+^*/(K_+^*)^{p^{n+1}}$ where $K_+ = K(\bar{\rho})$ for $\bar{\rho}$ a primitive p^{n+1}-st root of unity.)

Now observe that $\varinjlim Y'(n)$ is divisible, in fact isomorphic to a copy of $\mathbf{Z}(p^\infty)$. Hence the sequence in (4.3) splits, as does

$$0 \to \varinjlim Y'(n) \to X(F)_p \to X(F)_p \Big/ \varinjlim Y'(n) \to 0,$$

so we obtain

$$X(F(t))_p \cong X(F)_p \oplus \sum_f \varinjlim X(n, f). \tag{4.4}$$

Now we describe the $X(n, F)$. Let $\mathcal{A}_f = \{g_1, \ldots, g_r\}$ where $\sigma(g_1) = g_2, \sigma(g_2) = g_3, \ldots, \sigma(g_r) = g_1$. We have $s = ra$ for some a. The group $\langle g_1 \rangle \oplus \cdots \oplus \langle g_r \rangle$ is isomorphic to $\mathbf{Z}^r$ in the standard way, and under this isomorphism $X(n, f)$ can be described as follows: $X(n, f)$ is isomorphic to the subgroup of $\mathbf{Z}^r / q\mathbf{Z}^r$ generated by the images of the r-tuple

$$b_1 = (m^{s-1} + m^{s-r-1} + \cdots + m^{r-1}, m^{s-2} + m^{s-r-2} + \cdots + m^{r-2}, \ldots, m^{s-r} + m^{s-2r} + \cdots + 1)$$

and its right cyclic shifts $b_2, b_3, \ldots, b_r$. Now $mb_1 - b_2 = (m^s - 1, 0, 0, \ldots, 0)$, so the images $m\bar{b}_1$ and $\bar{b}_2$ are equal in $\mathbf{Z}^r / q\mathbf{Z}^r$, and similarly for $mb_2 - b_3$, etc. Hence $\bar{b}_1, \ldots, \bar{b}_r$ all generate the same cyclic subgroup of $\mathbf{Z}^r / q\mathbf{Z}^r$. Its order is q divided by the highest power of p dividing $m^{s-r} + m^{s-2r} + \cdots + 1 = (m^s - 1)/(m^r - 1)$ and hence, since $q = p^n$ exactly divides $m^s - 1$, we conclude that $X(n, f)$ is cyclic of order p^i where p^i exactly divides $m^r - 1$. This completes the description of $X(n, f)$ which, together with (4.4), gives the conclusion of Theorem 3.

We are now ready to produce an example of character groups with matching Ulm invariants which are not isomorphic. In our next example, the divisible parts will not match, but the reduced parts will provide an interesting example. We will produce an example in Theorem 6 for which all invariants match, the divisible parts included, but the character groups are still not isomorphic.

Let F_1 be the field K of Theorem 2; $K = \Pi_{\mathcal{F}} K_i$ where each K_i is the specified algebraic extension of $\mathbf{Q}(\varepsilon(8))$. Let F_2 be the field L_2 of Theorem 2; $L_2 = \mathbf{Q}_3(\varepsilon(8), t) = F(t)$, $F = \mathbf{Q}_3(\varepsilon(8))$. We set $R(X(F_i))$ to be the reduced group of the character group $X(F_i)$, $i = 1, 2$. We have:

THEOREM 4. *$R(X(F_i))$, $i = 1, 2$, have identical Ulm invaraints for all primes p, but $R(X(F_1))$ is not isomorphic to $R(X(F_2))$. In fact, for any prime p, $R(X(F_2))_p$ is a direct sum of cyclic groups, whereas $R(X(F_1))_p$ has no such decomposition.*

PROOF. We worked out the Ulm invariants of $X(F_1)$ in the course of proving Theorem 2. Using the function $f(p)$ of Theorem 2, these are given by:

(a) $U_p(\lambda, X(F_1)) = \mathscr{C}$ if $f(p) - 1 \leqq \lambda < \omega$,

(b) $U_p(\lambda, X(F_1)) = 0$ if $0 \leqq \lambda \leqq f(p) - 2$ or $\lambda \geqq \omega$.

We show now that the invariants of $X(F_2)$ match (a) and (b). We established during the proof of Theorem 2 that $U_p(\lambda, X(F_2)) = 0$ for $0 \leqq \lambda \leqq f(p) - 2$. Also, $RX(F_2)$ is a direct sum of cyclic groups by Theorem 3, so (b) is satisfied.

It remains to show that (a) holds. Let p be fixed, and take $n \geqq f(p) - 1$. We will show that F_2 has $\mathscr{C}$ independent cyclic extensions of dimension p corresponding to elements of order p and height n in $X(F_2)_p$. Let M be any finite extension of F containing a primitive q-th root of unity, $q = p^n$. M may be considered a residue class field of F_2 with respect to the discrete rank 1 valuation ν of F_2 corresponding to the irreducible polynomial $f(t) \in F[t]$, $f(t)$ the irreducible polynomial of any primitive element of M/F. As in the proof of Theorem 2, we construct the cyclic extension S_1 of M:

$$S_1 \supset S \supset M,$$

where S_1/M has dimension q, $[S: M] = p$, and any character for S/M has height n in $X(M)_p$.

Using Saltman's lifting theorem [8, 5.8], there is a cyclic extension $T_1 \supset F_2$ of dimension q so that ν is inert in T_1 and the residue class field of T_1 is S_1. Let T/F_2 be the layer of dimension p in T_1, and χ a corresponding character of order p in $X(F_2)_p$. By Lemma 2 of [4], χ has exact height n in $X(F_2)_p$. Since there are $\mathscr{C}$ independent choices for M and ν, there will be $\mathscr{C}$ independent extensions of this sort by the argument of [4, Theorem 19]. This shows that the Ulm invariants of $X(F_2)$ obey (a) above. (These invariants can also be deduced from the explicit calculation of the $X(n, f)$ in (4.3).) By (2) of Theorem 1 and (2.1), $RX(F_1)_p$ is not a direct sum of cyclic groups for any p. This concludes the proof of Theorem 4.

Note that the divisible subgroups of $X(F_1)_p$ and $X(F_2)_p$ do not match; in fact $r_p(X(F_1)_p) = \mathscr{C}$ while $r_p(X(F_2)_p) = 1$.

Our last example will provide two fields whose character groups have identical Ulm invariants, divisible parts included, but these character groups will be non-isomorphic.

LEMMA A. *Let F be a field, I an index set which is not necessarily countable, $F[x_i]_{i \in I}$ the polynomial ring, and $\mathscr{P}$ a partition of I such that each $c \in \mathscr{P}$ is finite. For each $c \in \mathscr{P}$ let $t_c = \Pi_{i \in c}\, x_i$. Then every irreducible polynomial in the polynomial ring $F[t_c]_{c \in \mathscr{P}}$ which is not a monomial remains irreducible in $F[x_i]_{i \in I}$.*

PROOF. Suppose, to simplify notation, we have $t = t_c = x_1 x_2 \cdots x_n$, $A = F[x_i]_{i \in I}$, and $B = F[t, 1/t]$. Then every irreducible element of $F[t]$ remains irreducible in B unless it becomes a unit, i.e. unless it is a monomial. Set $C = B[x_1, \ldots, x_{n-1}]$. Then $D = C[1/x_1, \ldots, 1/x_{n-1}]$ coincides with $A[1/x_1, \ldots, 1/x_n]$, i.e. the result of localizing C at $x_1, \ldots, x_{n-1}$ is the ring obtained by localizing A at all of $x_1, \ldots, x_n$. As C is a polynomial extension of B, an irreducible element of B remains irreducible in C; if such an element is not a monomial it will remain irreducible in the localization D. As D is also a localization of A, irreducible elements of D coming from A must have been irreducible in A. If we argue this way by adjoining all monomials t_c we get the desired result.

Now let F be a field, say $F = \mathbf{Q}$ or $\mathbf{Q}_3$, and let $\mathcal{S}$ be an index set of cardinality $\mathcal{C} = 2^{\aleph_0}$. For each $n \geqq 1$ let ρ_n be a primitive 3^n-th root of unity. For each $i \in \mathcal{S}$ and each $n \geqq 1$ let $x_i^{n,1}, x_i^{n,2}, \ldots, x_i^{n,\phi(3^n)}$ be indeterminates which we adjoin to $F(\rho_n)$ to obtain

$$E_n = F(\rho_n)(x_i^{n,k})_{i \in \mathcal{S},\ 1 \leqq k \leqq \phi(3^n)}$$

subject to the relations

$$(*) \qquad x_i^{n+1,k} x_i^{n+1,k+\phi(3^n)} x_i^{n+1,k+2\phi(3^n)} = x_i^{n,k}.$$

Let $E_\infty = \varinjlim E_n$.

Let σ be the automorphism of $F(\rho_n)$ which is the identity on F and maps ρ_n to ρ_n^2. We assume σ is the generator of the cyclic group $\mathrm{Gal}(F(\rho_n)/F)$ and that this group has order $\phi(3^n)$; this is valid if $F = \mathbf{Q}$ or $F = \mathbf{Q}_3$. Extend σ to each E_n by defining $\sigma(x_i^{n,k}) = x_i^{n,k+1}$ for each $k < \phi(3^n)$, $\sigma(x_i^{n,\phi(3^n)}) = x_i^{n,1}$. The relations $(*)$ guarantee that the defining relations of σ are compatible for different n, so σ can be considered as an automorphism of E_∞. For any polynomial f in the $x_i^{n,k}$, we also write f^σ for $\sigma(f)$.

For each n let F_n be the fixed field of E_n under σ, and let F_∞ be the fixed field of E_∞ under σ. Then we have

$$F_\infty = \varinjlim F_n.$$

THEOREM 5. *$X(F_\infty)_3 \cong X(F)_3 \oplus \mathbf{Z} \oplus \Sigma$ where $\mathbf{Z}$ is divisible of rank $\mathcal{C}$ and Σ is a direct sum of cyclic groups.*

PROOF. We have $X(F_\infty)_3 = \varinjlim X(F_n) = \varinjlim X(F_n)_3[3^n]$, where the latter denotes the elements of exponent 3^n in $X(F_n)_3$. Henceforth we write $X(n) = X(F_n)_3[3^n]$.

Now observe that $E_n = F_n(\rho_n)$. The Galois group of E_n/F_n is cyclic of order $s = \phi(3^n)$ generated by σ, and the results of [8] apply. We use the results of [8] to obtain compatible decompositions of $X(n)$.

Define $\Phi_n: E_n^* \to E_n^*$ by

$$\Phi_n(x) = x^{2^{s-1}} \cdot (x^{\sigma})^{2^{s-2}} \cdot (x^{\sigma^2})^{2^{s-3}} \cdots (x^{\sigma^{s-1}}).$$

Then it follows from [8, Theorem 2.3] (as in the proof of Theorem 3) that

$$X(n)/Y'(n) \cong \Phi(E_n^*) \cdot (E_n^*)^{3^n}/(E_n^*)^{3^n},$$

the subgroup of $E_n^*/(E_n^*)^{3^n}$ generated by images of elements of $\Phi(E_n^*)$. Here $Y'(n)$ denotes the subgroup of $X(n)$ corresponding to the subfield of $F(\rho_n)$ of dimension 3^{n-1} over F_n. Similarly, if $Y(n) = X(F)_3[3^n]$, we have

$$Y(n)/Y'(n) \cong \Phi(F(\rho_n)^*) \cdot (F(\rho_n)^*)^{3^n}/(F(\rho_n)^*)^{3^n}.$$

Now E_n is the quotient field of the polynomial ring $R_n = F(\rho_n)[x_i^{n,k}]_{i \in \mathcal{S},\, 1 \leqq k \leqq \phi(3^n)}$; unique factorization in R_n allows us to describe E_n^*. We choose one representative from each associate class of irreducible polynomials in R_n subject to the following restrictions:

(1) each $x_i^{n,k}$, $i \in \mathcal{S}$, $1 \leqq k \leqq \phi(3^n)$ is chosen.

(2) For each polynomial f chosen, if f^{σ^j} is an associate of f for some j, then $f^{\sigma^j} = f$. In this case $f^{\sigma}, f^{\sigma^2}, \ldots$, etc. will also be taken as representatives for their classes.

Condition (2) above requires some justification; suppose $f^{\sigma^j} = uf$ where $u \in F(\rho_n)$. Let $m = \operatorname{order}(\sigma^j)$, $\tau = \sigma^j$. Then iterating m times gives $f = f^{\tau^m} = u \cdot \tau(u) \cdot \tau^2(u) \cdots \tau^{m-1}(u) f = N_{F(\rho_n)/K}(u) f$, $K =$ fixed field of τ. Thus $N_{F(\rho_n)/K}(u) = 1$, so by Hilbert's Theorem 90 $u = \nu/\tau(\nu) = \nu/\sigma^j(\nu)$, some $\nu \in F(\rho_n)$. Now replacing f by νf gives $\tau(\nu f) = \nu f$, as desired.

Let $\mathcal{M}$ be the collection of σ-orbits of polynomials (which are not monomials) chosen in (2) above. Then we can write

$$(**) \qquad E_n^* = F(\rho_n)^* \oplus \sum_i \sum_k \langle x_i^{n,k} \rangle \oplus \sum_{M \in \mathcal{M}} \sum_{f \in M} \langle f \rangle.$$

Both Φ and "3^n-th power" respect the outer decomposition in (**), so we obtain the following decomposition of $X(n)/Y'(n)$:

$$(***) \qquad X(n)/Y'(n) = Y(n)/Y'(n) \oplus \sum_i Z_i(n) \oplus \sum_{M \in \mathcal{M}} X(n, M).$$

It follows immediately from [8, Theorem 2.3] that $Z_i(n)$ is cyclic of order 3^n.

Also, the proof of Theorem 3 gives $X(n, M)$ is cyclic of order 3^i where 3^i is the highest power of 3 dividing $2^{|M|}-1$.

To describe the direct limit of the $X(n)/Y'(n)$, observe that the natural map from $X(n)/Y'(n)$ to $X(n+1)/Y'(n+1)$ induces in (***) the obvious map on the first factor, and on the second and third factors the map induced by the "cubing" map from $E_n^*/(E_n^*)^{3^n}$ to $E_{n+1}^*/(E_{n+1}^*)^{3^{n+1}}$. This respects the decomposition in the second factor, and embeds $Z_i(n)$ into $Z_i(n+1)$. Thus the $Z_i(n)$ generate a copy of $\mathbf{Z}(3^\infty)$ in the limit, and there are $\mathscr{C}$ such copies generated as $\mathscr{S}$ has cardinality $\mathscr{C}$.

We now consider the third factor of (***). By Lemma A, any $f \in M$ remains irreducible in R_{n+1} as long as it remains irreducible in $R_n(\rho_{n+1})$. Such an f might split in $R_n(\rho_{n+1})$ and even further at later stages, but such splitting will eventually terminate (as can be seen by factoring f in $R_n(\rho_\infty)$, the ring obtained by adjoining all 3-power roots of unity to R_n). Hence we obtain:

$$\varinjlim X(n)/Y'(n) \cong \varinjlim Y(n)/Y'(n) \oplus \sum_i Z_i \oplus \Sigma$$

where each Z_i is a copy of $\mathbf{Z}(3^\infty)$, and Σ is a direct sum of cyclic groups. Now, arguing as in Theorem 3, we get:

$$X(F_\infty)_3 \cong X(F_3) \oplus \mathbf{Z} \oplus \Sigma$$

where $\mathbf{Z} = \Sigma_{i \in \mathscr{S}} Z_i$ is a divisible 3-group of rank $2^{\aleph_0}$ and Σ is a direct sum of cyclic groups.

This completes the proof of Theorem 5.

Remark. For the application to follow, we will need to know the precise structure of Σ. There is one cyclic factor in Σ for each σ-orbit of associate classes of irreducibles (excluding monomials) in $\varinjlim R_n$, and this factor has order 3^j where 3^j exactly divides $2^a - 1$ and a is the size of the orbit. Since orbits of all sizes exist and $\mathscr{S}$ has cardinality $\mathscr{C} = 2^{\aleph_0}$, we conclude that Σ has $2^{\aleph_0}$ factors of each 3-power order.

We are now ready to construct our last example. Let F be the field obtained from the 3-adic field $\mathbf{Q}_3$ by adjoining all p^n-th roots of unity for all n and all primes $p \neq 3$. Using F as a base field we construct the field $L = F_\infty$ as in Theorem 5. Note that the results of Theorem 5 apply since L has no primitive cube root of unity, and adjoining 3^n-th roots of unity to L produces an extension of L of degree $\phi(3^n)$.

Our field M will be an ultraproduct. We begin with the rational field $\mathbf{Q}$, and close it under the following construction: for any prime $p \neq 3$, the p^n-th roots of unity of $\mathbf{Q}$ produce an extension of degree $(p-1)p^{n-1}$; we adjoin the corresponding extension of degree p^{n-1} to $\mathbf{Q}$ (for all n). Let the resulting field be k, and note that k is of finite codimension in a field containing all p^n-th roots of unity, $p \neq 3$. Let M be a non-principal ultraproduct of countably many copies of k. We have:

THEOREM 6. *For all primes p, $X(L)_p$ and $X(M)_p$ have identical Ulm invariants, including their divisible parts. These groups are divisible and isomorphic if $p \neq 3$, but $X(L)_3$ and $X(M)_3$ are not isomorphic.*

PROOF. First suppose $p \neq 3$. Since L contains all p^n-th roots of 1, $X(L)_p$ is divisible. M is of finite codimension in a field containing all p^n-th roots of 1, and the codimension (in this case $p-1$) is prime to p. The restriction–corestriction argument of [1, Theorem 3] shows then that $X(M)_p$ is divisible. It is clear by inspection that $X(L)_p$ and $X(M)_p$ are direct sums of $\mathscr{C}$ copies of $\mathbf{Z}(p^\infty)$, so these groups are isomorphic.

By Theorem 5, $X(L)_3 \cong X(F)_3 \oplus D \oplus \Sigma$ where D is a direct sum of $\mathscr{C}$ copies of $\mathbf{Z}(3^\infty)$, Σ is a direct sum of $\mathscr{C}$ copies of $\mathbf{Z}/3^j\mathbf{Z}$, $j = 1,2,3,\ldots$, and F is the field obtained by adjoining all p^n-th roots of 1 to $\mathbf{Q}_3$, $p \neq 3$. We claim $X(F)_3$ is divisible. To see this, observe that F is a direct limit of countably many fields E such that $[E:\mathbf{Q}_3] = m < \infty$ and E has no primitive cube root of unity. By [9, Theorem 3, p. II-30], the Galois group of the maximal 3-extension of E is free pro-3 on $m+1$ generators, so its dual $X(E)_3$ is a finite direct sum of copies of $\mathbf{Z}(3^\infty)$. We conclude that $X(F)_3$ is divisible and, in fact, countable. Thus as abstract groups, $X(L)_3 \cong D \oplus \Sigma$; its divisible rank is $\mathscr{C}$ and its Ulm invariants $=$ $\mathscr{C}$ at all finite ordinals. For $X(M)_3$, it is clear that its divisible rank is also $\mathscr{C}$. We claim the invariants for $X(M)_3$ are $\mathscr{C}$ at all finite ordinals. To see this, we note that the Ulm invariants of $X(\mathbf{Q})_3$ are ω at all finite ordinals as observed in [1]–[4]. Since k is obtained from $\mathbf{Q}$ by a sequence of extensions of degree prime to 3, the argument of [1, Theorem 3] shows that $X(k)_3$ has Ulm invariant $= \omega$ at all finite ordinals. By (2.1), $X(M)_3$ has invariant $= \mathscr{C}$ at all finite ordinals.

We now have that $X(L)_p$ and $X(M)_p$ have identical invariants for all p. However, $X(M)_3$ is not a direct sum of cyclic and divisible groups by Theorem 1, so $X(M)_3$ is not isomorphic to $X(L)_3$.

We have included both Theorems 4 and 6 because they are somewhat dual in nature. In Theorem 4 our two fields have character groups which have non-trivial reduced groups for all primes p; those reduced groups have identical Ulm invariants but are non-isomorphic for all p. In Theorem 4, the divisible parts of the character groups differ for all p. In Theorem 6, the p-components of the character groups are divisible and isomorphic for all $p \neq 3$; the reduced groups are non-trivial only for $p = 3$, have identical Ulm invariants, but are not isomorphic. The divisible components in Theorem 6 also match for $p = 3$.

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